PROCEEDINGS

OF

THE ROYAL SOCIETY.

December 11, 1890.

Sir WILLIAM THOMSON, D.C.L., LL.D., President, in the Chair.

The President announced that he had appointed as Vice-Presidents—

The Treasurer.
The Astronomer Royal.
Professor Alfred Newton.
Sir G. Gabriel Stokes.
Lieut.-General Strachey.

The Presents received were laid on the table, and thanks ordered for them.

The following Papers were read :—

I. "On Ellipsoidal Harmonics." By W. D. NIVEN, F.R.S. Received October 23, 1890.

(Abstract.)

In the paper, of which the following is an abstract containing statements of the principal results arrived at, an attempt has been made to develop the subject of ellipsoidal harmonics from their expressions in Cartesian coordinates.

The harmonics of the ellipsoid of three unequal axes are first investigated, as being the most readily dealt with on account of symmetry and, afterwards, those of the prolate and oblate spheroids are deduced as particular cases.

1. It has been found convenient to discuss separately the forms which are respectively suitable to the inside and outside of the ellipsoid, the former being taken first—

If Θ_r denote $$\frac{x^2}{a^2+\theta_r}+\frac{y^2}{b^2+\theta_r}+\frac{z^2}{c^2+\theta_r}-1,$$

the expressions comprised under

$$G = (1,\ x,\ y,\ z,\ yz,\ zx,\ xy,\ xyz)\ \Theta_1 \dots . \Theta_n,$$

where any of the quantities inside the brackets is the multiplier of the product of the Θ's outside, will satisfy Laplace's equation, provided n equations of the form

$$\frac{p}{a^2+\theta_1}+\frac{q}{b^2+\theta_1}+\frac{r}{c^2+\theta_1}+\frac{4}{\theta_1-\theta_2}+\dots . +\frac{4}{\theta_1-\theta_n}=0$$

are satisfied, where p, q, r are respectively 3 or 1 according as G does or does not contain x, y, z as factors.

2. If K_r denote $\frac{x^2}{a^2+\theta_r}+\frac{y^2}{b^2+\theta_r}+\frac{z^2}{c^2+\theta_v}$, so that $K_r = \Theta_r + I$, then, in like manner, the expressions comprised under

$$H = (1,\ x,\ y,\ z,\ yz,\ zx,\ xy,\ xyz)\ K_1 \dots . K_n$$

will satisfy Laplace's equation for precisely the same values of θ as in §1, and it may be shown that there are $2n+1$ independent conjugate H-harmonics of any degree n.

3. The function H is a spherical harmonic. Suppose it is of the nth degree and of order σ, and let it be denoted by H_n^{σ}. The corresponding ellipsoidal harmonic, *i.e.*, for the same values of θ, may be denoted by G_n^{σ}, and it may be shown that G_n^{σ} and H_n^{σ} are connected by the relation

$$G_{n\sigma} = \Big\{1-\frac{D^2}{2(2n-1)}+\dots . \\ +(-1)^r\frac{D^{2r}}{2^r r!\,(2n-1)\,(2n-3)\dots .(2n-2r+1)}+\dots . \Big\}H_n^{\sigma},$$

where $$D^2 = a^2\frac{\partial^2}{\partial x}+b^2\frac{\partial^2}{\partial y}+c^2\frac{\partial^2}{\partial z}.$$

4. Let xyz be any point on the surface of the ellipsoid and $x'y'z'$ the corresponding point on a concentric sphere of unit radius, so that

$$x = ax', \qquad y = by', \qquad z = cz',$$

then will $$\Theta_r\,(x,\,y,\,z) = -\theta_r K_r\,(x',\,y',\,z'),$$

and

$$G\,(x,\,y,\,z) = (1,\,a,\,b,\,\dots .\,,\,abc)\,(-\theta_1)\,(-\theta_2)\dots .\,H\,(x',y',z').$$

By means of these relations any function $f\,(x,\,y,\,z)$ or $f\,(ax',\,by',\,cz')$

can be first expressed in terms of spherical harmonics in x', y', z', by Laplace's expansion, and then in ellipsoidal harmonics in x, y, z.

A series of ellipsoidal harmonics can thus be found having an arbitrary value at the surface of the ellipsoid.

5. *External Harmonics.*—The leading proposition in this part of the subject is as follows :—

If $\pi abc V_n$ denote the potential at an outside point xyz due to a solid ellipsoid, whose semi-axes are a, b, c, such that the density at any internal point fgh is of the form

$$n\left(1-\frac{f^2}{a^2}-\frac{g^2}{b^2}-\frac{h^2}{c^2}\right)^{n-1},$$

then the harmonic of degree n and order σ, suitable to the space outside of the ellipsoid, is given by

$$G_n^\sigma I_n^\sigma = (-1)^n \frac{1}{2^n n!} H_n^\sigma \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) V_n,$$

where $$I_n^\sigma = \int_\epsilon^\infty \frac{d\lambda}{(\theta_1-\lambda)^2 \ldots . \sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}.$$

This result may primarily be regarded as a means of reducing the integral on the left-hand side, when the values of θ are known, into simpler forms, which can be actually evaluated when the surface is one of revolution. It is a result of some importance in the subject, as containing within itself the numerous expressions into which the external harmonics of spheroids can be thrown.

6. *Spheroids.*—The foregoing formulæ admit of easy reduction when two of the axes of the ellipsoid are equal, say $a = b$. It may then be shown that the spherical harmonics H of §2 are the ordinary spherical harmonic conjugate system. It is therefore convenient to adopt the definitions and specifications in Thomson and Tait's 'Natural Philosophy,' and thus to harmonise the spheroidal system with the spherical. Accordingly, if H be now used to express a spherical harmonic according to the definitions in that work, the new signification of G will be in accordance with the relation in § 3.

Taking the results contained in §§ 3, 5, and effecting reductions suitable to the prolate spheroid, we obtain the following :—

$$G_n^\sigma = j^\sigma \frac{2^{n-\sigma}(n+\sigma)!\,(n-\sigma)!}{(2n)!} \gamma^n . 2\cos\sigma\phi . \frac{1}{\pi}\int_0^\pi P_n\left(\frac{z+j\rho\cos\theta}{\gamma}\right)\cos\sigma\theta\, d\theta,$$

$$I_n^\sigma = \frac{(2n)!\,(2n)!}{2^{3n}(n!)^3(n-\sigma)!} \frac{1}{\gamma^{2n+1}\dfrac{d^\sigma}{d\mu^\sigma}P_n(\mu)} \int_{-1}^1 \frac{(1-v^2)^n}{(\mu-v)^{n+\sigma+1}}\,dv,$$

where $$\gamma^2+\epsilon = \mu^2\gamma^2, \qquad \gamma^2 = c^2-a^2.$$

From these forms for G_n^σ and $G_n^\sigma I_n^\sigma$ a variety of others may be obtained, as well as expressions in the form of integrals for spherical harmonics of the second kind. Corresponding forms may also be established for oblate spheroids.

7. The expansion of the reciprocal of the distance between two points plays an important part in the application of these investigations. It has therefore been found in ellipsoidal harmonics and thence, by reduction, in harmonics of the spheroid, circular cylinder, and paraboloid of revolution, and its application has been briefly illustrated in finding the general term in the expansion of the potential due to the magnetism induced in an ellipsoid placed in any field of force, and in finding the electrical capacities of surfaces inverted from ellipsoids. In the same connexion, I have also found the expansion for the potential due to a thin shell bounded by similar and similarly situated ellipsoids, the density of which varies inversely as the cube of the distance from a fixed point.

8. In the last part of the paper I have shown how to prove what Heine terms "addition theorems" in the case of spheroidal harmonics, and thence, by reduction, in the case of Bessel's functions.

II. "Photometric Observations of the Sun and Sky." By William Brennand. Communicated by C. B. Clarke, F.R.S. Received October 30, 1890.

(Abstract.)

1. The paper begins with a short account of the various papers communicated by Sir H. Roscoe, and published in the Transactions of the Royal Society.

2. My observations were made at Dacca, East Bengal, in 1861–66, repeated at Milverton, in Somersetshire, during the last two years. My first experiments were directed to ascertaining the action of the sun on sensitised paper exposed at right angles to the solar rays for different altitudes of the sun, and largely to ascertaining the laws of distribution of the actinic power in the sky.

I take no observations except when the sky is quite clear.

3. The method of measurement I adopted is the darkening produced in sensitised paper. I cut strips from one uniform sheet of ordinary photographic paper. My observations being relative, I obtain the same results (ratios) with any paper. I compare ultimately the effects of the sun and of a candle on this same paper.

4. I assume that, in burning a stearine candle, the chemical action is proportional to the material consumed; I have taken as my unit (i)